Correlations in scale-free networks: Tomography and percolation

R. Xulvi-Brunet, W. Pietsch, and I. M. Sokolov
Institut für Physik, Humboldt Universität zu Berlin, Newtonstraße 15, D-12489 Berlin, Germany
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We discuss three related models of scale-free networks with the same degree distribution but different correlation properties. Starting from the Barabási-Albert construction based on growth and preferential attachment we discuss two other networks emerging when randomizing it with respect to links or nodes. We point out that the Barabási-Albert model displays dissortative behavior with respect to the nodes' degrees, while the node-randomized network shows assortative mixing. These kinds of correlations are visualized by discussing the shell structure of the networks around an arbitrary node. In spite of different correlation behaviors, all three constructions exhibit similar percolation properties. This result for percolation is also detected for a network with finite second moment and its corresponding randomized models.

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INTRODUCTION

Scale-free networks, i.e., networks with power-law degree distributions, have recently been widely studied (see Refs. [1,2] for a review). Such degree distributions have been found in many different contexts, for example, in several technological webs such as the Internet [3,4], the World Wide Web [5,6], or electrical power grids [7], in natural networks such as the network of chemical reactions in the living cell [8–10], and also in social networks such as the network of human sexual contacts [11], the science [12,13] and the movie actor [14,15] collaboration networks, or the network of the phone calls [16].

The topology of networks is essential for the spread of information or infections, as well as for the robustness of networks against intentional attack or random breakdown of elements. Recent studies have focused on a more detailed topological characterization of networks, in particular, in the degree correlations among nodes [4,17–26]. For instance, many technological and biological networks show that nodes with high degree connect preferably to nodes with low degree [4,21], a property referred to as dissortative mixing. On the other hand, social networks show assortative mixing [17,25], i.e., highly connected nodes are preferably connected to nodes with high degree.

In this paper we shall study some aspects of this topology, specifically the importance of the degree correlations in scale-free networks and concentrate on the two following important characteristics: the tomography of shell structure around an arbitrary node and percolation. We will introduce a procedure to change correlations in networks that produces assortative mixing. We shall compare the correlation properties of this model with the classical Barabási-Albert construction [1,27] and with an uncorrelated model. Although some results are already known in the literature we corroborate these findings and compare them with the results of our model.

THE MODELS

Our starting model is the one of Barabási and Albert (BA) [27], based on the growth algorithm with preferential attachment. Starting from an arbitrary set of initial nodes, at each time step a new node is added to the network. This node brings with it m proper links which are connected to m nodes already present. The latter are chosen according to the preferential attachment prescription: The probability that a new link connects to a certain node is proportional to the degree (number of links) of that node. The resulting degree distribution of such networks tends to [28–30]

$$P(k)=\frac{2m(m+1)}{k(k+1)(k+2)}\sim k^{-3}. \tag{1}$$

Krapivsky and Redner [30] have shown that in the BA construction correlations develop spontaneously between the degrees of connected nodes. To assess the role of such correlations we shall randomize the BA network.

Recently, Maslov and Sneppen [21] have suggested an algorithm randomizing a given network that keeps the degree distribution constant. According to this algorithm at each step two links of the network are chosen at random. Then, one end of each link is selected randomly and the attaching nodes are interchanged. In case one, or both, of these new links already exists in the network, this step is discarded and a new pair of edges is selected. This restriction prevents the appearance of multiple edges connecting the same pair of nodes. A repeated application of the rewiring step leads to a randomized version of the original network. We shall refer to this model as the link-randomized (LR) model.

The LR model can be compared with another model which is widely studied in the context of scale-free networks, namely, with the configuration model introduced by Bender and Canfield [31,32]. It starts with a given number N of nodes. Then is assigned to each node a number k_i of ''edge stubs'' equal to its desired connectivity. The stubs of different nodes are then connected randomly to each other; two connected stubs form a link. One of the limitations of this ''stub reconnection'' algorithm is that for broad distribution of connectivities, which is usually the case in complex networks, the algorithm generates multiple edges joining the same pair

of hub nodes and loops connecting the node to itself. However, the configuration model and the LR model get equivalent as $N \to \infty$.

One can modify the link-randomization procedure in such a way that it generates an assortative network. Instead of choosing randomly two links, we now choose uniformly at random two nodes in the network. Of each node, we then select randomly one link. Taking these two links we continue with the link-randomization procedure as above. We call the resulting networks node randomized (NR).

As we proceed to show, the three models have different properties with respect to the correlations between the degrees of connected nodes. While the LR (configuration) model is random, the genuine BA prescription leads to a network which is dissortative with respect to the degrees of connected nodes and the NR model leads to an assortative network. This fact leads to considerable differences in the shell structure of the networks and also to some (not extremely large) differences in their percolation characteristics. We hasten to note that our simple models neglect many important aspects of real networks such as geography [33,34] but stress on the importance to consider the higher correlations in the degrees of connected nodes.

TOMOGRAPHY OF THE NETWORKS

Referring to spreading of computer viruses or human diseases, it is necessary to know how many sites get infected on each step of the infection propagation. Thus, we examine the local structure in the network. Cohen *et al.* [35] examined the shells around the node with the highest degree for uncorrelated networks. We will also examine the tomography for all our three models: the BA, the LR, and the NR model. However, in our study we start from a node chosen at random. This initial node (the root) is assigned to shell number 0. Then all links starting at this node are followed. All nodes reached are assigned to shell number 1. Then all links leaving a node in shell 1 are followed and all nodes reached that do not belong to previous shells are labeled as nodes of shell 2. The same is carried out for shell 2, etc., until the whole network is exhausted. We then get $N_{l,r}$, the number of nodes in shell l for root r. The whole procedure is repeated starting at all N nodes in the network, giving $P_l(k)$, the degree distribution in shell l. We define $P_l(k)$ as

$$P_l(k) = \frac{\sum_r N_{l,r}(k)}{\sum_{k,r} N_{l,r}(k)}. \tag{2}$$

We are most interested in the average degree $\langle k \rangle_l = \Sigma_k k P_l(k)$ of nodes of the shell l. In the epidemiological context, this quantity can be interpreted as a disease multiplication factor after l steps of propagation. It describes how many neighbors a node can infect on average. Note that such a definition of $P_l(k)$ gives us for the degree distribution in the first shell:

$$P_1(k) = \frac{\sum_r N_{1,r}(k)}{\sum_{k,r} N_{1,r}(k)} = \frac{kN_k}{\sum_k kN_k} = \frac{kP(k)}{\langle k \rangle}, \tag{3}$$

where $P(k)$ and N_k are the degree distribution and the number of nodes with degree k in the network, respectively. We bear in mind that every link in the network is followed exactly once in each direction. Hence, we find that every node with degree k is counted exactly k times. From Eq. (3) follows that $\langle k \rangle_1 = \langle k^2 \rangle / \langle k \rangle$. This quantity, which plays a very important role in the percolation theory of networks [36], depends only on the first and the second moment of the degree distribution, but not on the correlations. Of course $P_0(k) = P(k)$.

Note that as $N \to \infty$ we have $\langle k^2 \rangle \to \infty$: for our scale-free constructions the mean degree in shell 1 depends significantly on the network size determining the cutoff in the degree distribution. For a given network size N the values of $\langle k \rangle_1$ are the same for all three models. The first two shells are determined only by the degree distributions. In all other shells the three models differ. For the LR (configuration) model one finds for all shells in the thermodynamic limit $P_l(k) = P_1(k)$. However, since these distributions do not possess finite means, the values of $\langle k \rangle_l$ are governed by the finite-size cutoff, which is different in different shells, since the network is practically exhausted within the first few steps, see Fig. 1.

In our simulations we use networks based on the BA construction with $m = 2$. For larger m the same qualitative results were observed. In the present work we refrain from discussion of a peculiar case $m = 1$. For $m = 1$ the topology of the BA model is distinct from the one for $m \geqslant 2$ since in this case the network is a tree. This connected tree is destroyed by the randomization procedure and is transformed into a set of disconnected clusters. On the other hand, for $m \geqslant 2$ the creation of large separate clusters under randomization is rather improbable, so that most of the nodes stay connected. Figure 1 shows $\langle k \rangle$ as a function of the shell number l. Panel (a) corresponds to the BA model, panel (b) to the LR model, and panel (c) to the NR model. The different curves show simulations for different network sizes: $N = 3\,000$, $N = 10\,000$, $N = 30\,000$, and $N = 100\,000$. All points are averaged over ten different realizations except for those for networks of 100 000 nodes with only one simulation. In panel (d) we compare the shell structure for all three models at $N = 30\,000$. The most significant feature of the graphs is the difference in $\langle k \rangle_2$. In the BA and LR models the maximum is reached in the first shell, while for the NR model the maximum is reached only in the second shell: $\langle k \rangle_{2,BA} < \langle k \rangle_{2,LR} < \langle k \rangle_{2,NR}$. This effect becomes more pronounced with increasing network size. In shells with large l for all networks mostly nodes with the lowest degree 2 are found.

The inset in graph (a) of Fig. 1 shows the relation between average age η of nodes with connectivity k in the network as a function of their degree for the BA model. The age of a node n and of any of its proper links is defined as

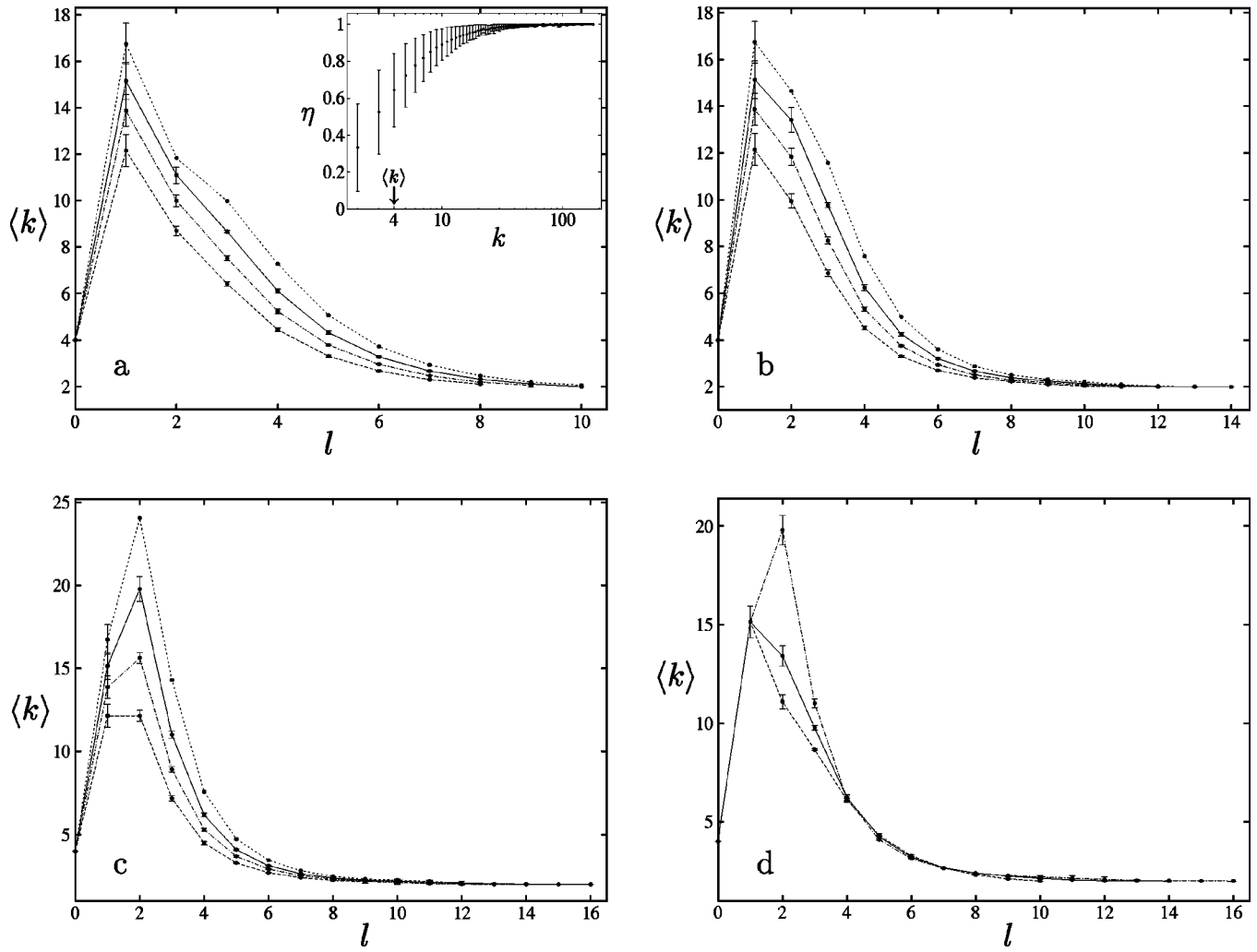

FIG. 1. Mean degree value $\langle k\rangle$ in shell l: (a) for the BA model, (b) for the LR model, (c) for the NR model. Different curves correspond to different network sizes: from top to bottom 100 000, 30 000, 10 000, and 3 000 nodes. Ten simulations were done for each value except for the shells with $l \geqslant 2$ at $N=100\,000$ based on only one simulation. Panel (d) compares the tomography of the models with $N=30\,000$: from top to bottom NR model, LR model, and BA model. The inset in panel (a) shows the average age η of a node as a function of its degree k.

$\eta(n)=(N-t_n)/N$, where t_n denotes the time of birth of the node. For the randomized LR and NR models age has no meaning. The figure shows a strong correlation between age and degree of a node. The reasons for these strong correlations are as follows. First, older nodes experienced more time steps than younger ones and thus have larger probability to acquire nonproper bonds. Moreover, at earlier times there are less nodes in the network, so that the probability of acquiring a new link per time step for an individual node is even higher. Third, at later time steps older nodes already tend to have higher degrees than younger ones, so the probability for them to acquire new links is considerably larger due to preferential attachment. The correlations between the age and the degree bring some nontrivial aspects into the BA model based on growth, which are erased when randomizing the network.

Let us discuss the degree distribution in the second shell. In this case we find that every link leaving a node of degree k in shell 1 is counted $k-1$ times. Let $P(l|k)$ be a probability that a link leaving a node of degree k enters a node with degree l. Neglecting the possibility of short loops (which is always appropriate in the thermodynamic limit $N\to\infty$) and the inherent direction of links (which may be not totally appropriate for the BA model) we have

$$P_2(l)=\frac{\sum_k kP(k)(k-1)P(l|k)}{\sum_k kP(k)(k-1)}. \tag{4}$$

The value of $\langle k\rangle_2$ gives important information about the type of mixing in the network. To study mixing in networks one needs to divide the nodes into groups with identical properties. The only relevant characteristic of the nodes that is present in all three models is their degree. Thus, we can examine the degree correlations between neighboring nodes, which we compare with the uncorrelated LR model, where

the probability that a link connects to a node with a certain degree is independent from whatever is attached to the other end of the link: $P(k|l)=kP(k)/\langle k\rangle=kP(k)/2m$. All other relations would correspond to assortative or dissortative mixing. Qualitatively, assortativity then means that nodes attach to nodes with similar degree more likely than in the LR model: $P(k|l)>P(k|l)_{LR}=kP(k)/\langle k\rangle$ for $k\approx l$. Dissortativity means that nodes attach to nodes with very different degree more likely than in the LR model: $P(k|l)>kP(k)/\langle k\rangle$ for $k\gg l$ or $l\gg k$. Inserting this in Eq. (4), and calculating the mean, one finds qualitatively that $\langle k\rangle_1=\langle k\rangle_{2,LR}<\langle k\rangle_2$ for assortativity and $\langle k\rangle_1>\langle k\rangle_2$ for dissortativity.

In the following we show where the correlations of the BA and NR models originate. A consequence of the BA algorithm is that there are two different types of ends for the links. Each node has exactly m proper links attached to it at the moment of its birth and a certain number of links that are attached later. Since each node receives the same number of links at its birth, towards the proper nodes a link encounters a node with degree k with probability $P(k)$. To compensate for this, in the other direction a node with degree k is encountered with the probability $(k-m)P(k)/m=2kP(k)/\langle k\rangle-P(k)$, so that both distributions together yield $kP(k)/\langle k\rangle$. On one end of the link, nodes with small degree are predominant: $P(k)<kP(k)/\langle k\rangle$ for small k. On the other end, nodes with high degree are predominant: $(k-m)P(k)/m>kP(k)/2m$ for k large. This corresponds to dissortativity. Actually the situation is somewhat more complex since in the BA model these probability distributions also depend on the age of the link.

Assortativity of the NR model is a result of the node-randomizing process. Since the nodes with smaller degree are predominant in the node population, those links are preferably chosen that have on the end, with the randomly chosen node, a node with a smaller degree [$P(k)>kP(k)/\langle k\rangle$ for k small]. Then the randomization algorithm exchanges the links and connects these nodes to each other. This leads to assortativity for nodes with small degree, which is compensated by assortativity for nodes with high degree.

PERCOLATION

Percolation properties of networks are relevant when discussing their vulnerability to attack or immunization which removes nodes or links from the network. For scale-free networks random percolation as well as vulnerability to a deliberate attack have been studied by several groups [36–40]. One considers the removal of a certain fraction of edges or nodes in a network. Our simulations correspond to the node removal model; q is the fraction of removed nodes. Below the percolation threshold $q<q_c$ a giant component (infinite cluster) exists, which ceases to exist above the threshold. A giant component, and consequently q_c, is exactly defined only in the thermodynamic limit $N\to\infty$: it is a cluster to which a nonzero fraction of all nodes belongs.

In Refs. [32,36] a condition for the percolation transition in random networks has been discussed: Every node already connected to the spanning cluster is connected to at least one new node. Reference [36] gives the following percolation criterion for the configuration model:

$$1-q_c=\frac{\langle k\rangle}{\langle k^2\rangle-\langle k\rangle}, \tag{5}$$

where the means correspond to an unperturbed network ($q=0$). For networks with degree distribution, Eq. (1), $\langle k^2\rangle$ diverges as $N\to\infty$. This yields for the random networks with such a degree distribution a percolation threshold $q_c=1$ in the thermodynamic limit, independent of the minimal degree m; in the epidemiological terms this corresponds to the absence of herd immunities in such systems. Crucial for this threshold is the power-law tail of the degree distribution with an exponent $\leqslant 3$. Moreover, Ref. [37] shows that the critical exponent β governing the fraction of nodes M_∞ of the giant component, $M_\infty\propto(q_c-q)^\beta$, diverges as the exponent of the degree distribution approaches -3. Therefore M_∞ approaches zero with zero slope as $q\to 1$.

In Fig. 2 we plotted for the three models discussed M_∞ as a function of q. The behavior of all three models for a network size of 300 000 nodes is presented in panel (a). In the inset the size of the giant component was measured in relation to the number of nodes remaining in the network $(1-q)N$ and not to their initial number N. The other panels show the percolation behavior of each of the models at different network sizes: Panel (b) corresponds to the BA model, panel (c) to the LR model, and panel (d) to the NR model. For the largest networks with $N=300\,000$ nodes we calculated five realizations for each model, and for those with 30 000, 10 000, and 3 000 nodes averaging over ten realizations was performed. For all three models within the error bars the curves at different network sizes coincide. This shows that even the smallest network is already close to the thermodynamic limit. Albert *et al.* found a similar behavior in a study of BA networks [38]. They analyze networks of sizes $N=1\,000$, 5 000, and 20 000 concluding that ''the overall clustering scenario and the value of the critical point is independent of the size of the system.''

In the simulations we find two regimes: for moderate q we find that the sizes of the giant components of the BA, LR, and NR models obey the inequalities $M_{\infty,BA}>M_{\infty,LR}>M_{\infty,NR}$, while for q close to unity the inequalities are reverted: $M_{\infty,BA}<M_{\infty,LR}<M_{\infty,NR}$. However, in this regime the differences between $M_{\infty,BA}$, $M_{\infty,LR}$, and $M_{\infty,NR}$ are subtle and hardly resolved on the scales of Fig. 2. We note that a similar situation was observed in Ref. [17]. However, there the size of the giant cluster was measured not as a function of q but of a scaling parameter in the degree distribution.

The observed effects can be explained by the correlations in the network. For $q=0$ one has $M_{\infty,BA}=M_{\infty,LR}=M_{\infty,NR}$. Now, the probability that single nodes lose their connection to the giant cluster depends only on the degree distribution and not on correlations. So, the difference in the M_∞ must be explained by the breakoff of clusters containing more than one node. The probability for such an event is smaller in the BA than in the LR model, since dissortativity implies that one finds fewer ''regions,'' where only nodes with low degree are present.

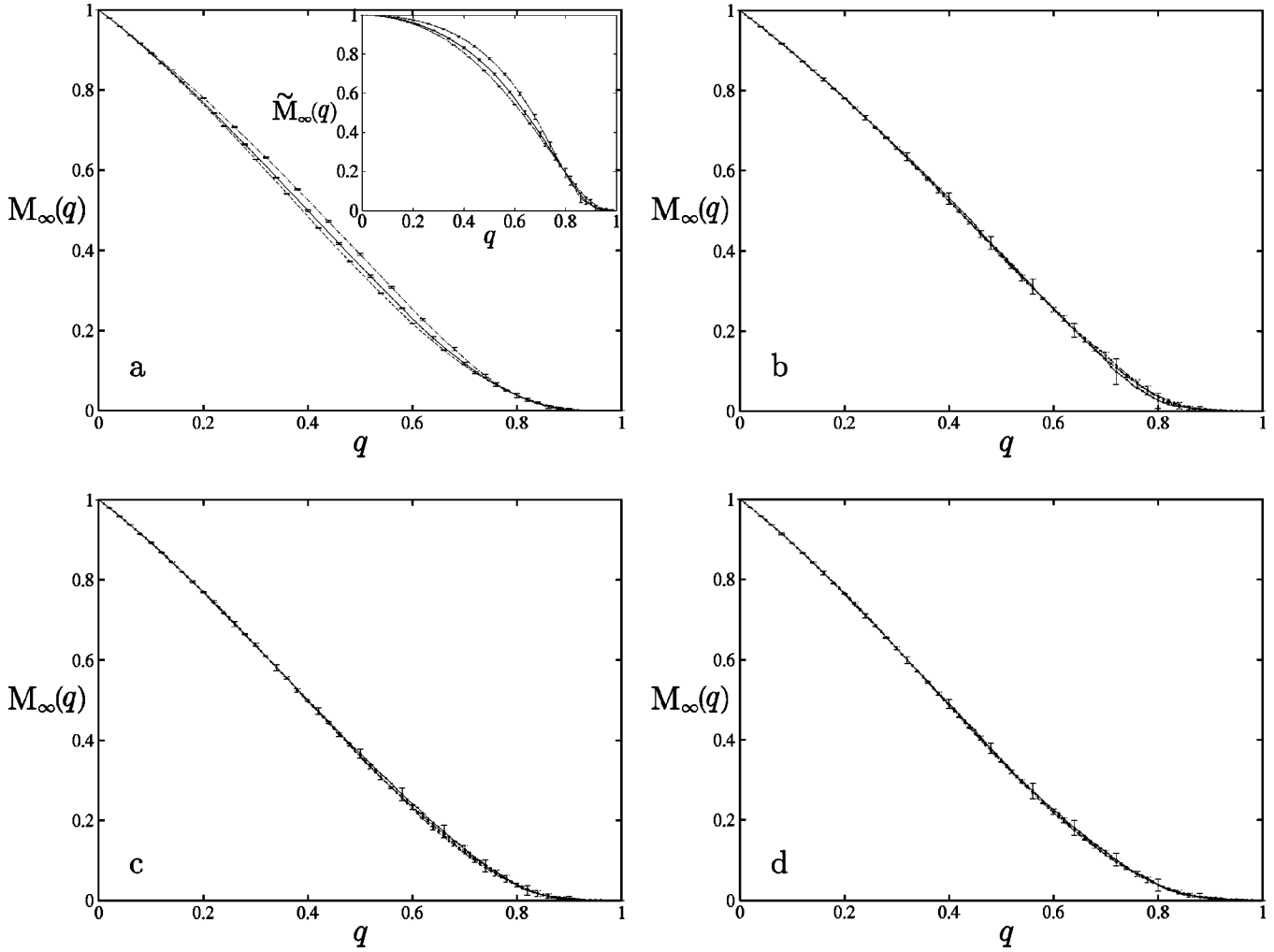

FIG. 2. Fraction of nodes M_∞ in the giant component depending on the fraction q of nodes removed from the network: (b) for the BA model, (c) for the LR model, and (d) for the NR model. Different curves correspond to different network sizes: from top to bottom 300 000 (five simulations), 30 000, 10 000, and 3 000 nodes (ten simulations each). Graph (a) compares all three models at $N=300\,000$ (from top to bottom: BA model, LR model, and NR model). The inset shows the fraction $\tilde{M}_\infty$ of the number of nodes in the giant component relative to the remaining number of nodes in the network $(1-q)N$.

However, when we get to the region of large q, as nodes with low degree act as ''bridges'' between the nodes with high degree, the connections between the nodes with high degree are weaker in the case of the BA model than in the case of the LR model. So, the probability that nodes with high degree break off is higher for the BA model than for the LR model. There is no robust core of high-degree nodes in the network [17]. The correlation effects for the NR model, when compared with the LR model, are opposite to those for the BA model.

Vázquez and Moreno proposed that some networks with assortative correlations may not exhibit a percolation transition even when the second moment of the degree distribution is finite [22]. We check their proposition applying our procedure to obtain assortative networks with a given degree distribution. For this purpose, we generate networks with finite second moment as described in Refs. [30,41]. We use the BA construction, but employ a different preferential attachment: the probability that a new link connects to a certain node is proportional to its connectivity plus a positive constant A, the initial attractiveness. The resulting degree distribution of such networks then tends to $P(k)\sim k^{-3-A/m}$ [30,41]. For example, choosing $A=4$ and $m=2$, the degree distribution tends to

$$P(k)=\frac{12\,096}{(k+4)(k+5)(k+6)(k+7)(k+8)}, \qquad (6)$$

which has the second moment $\langle k^2\rangle=28$.

Applying the NR and LR procedures to such a network with $N=10^5$ nodes we obtain three differently correlated systems. The results for percolation in these three cases are shown in Fig. 3. All points are averaged over ten different realizations. The upper curve corresponds to the initial network generated following the algorithm [30,41], the central curve corresponds to the same network after link randomization, and the lower curve after node randomization. No significant difference between the three models is detected in

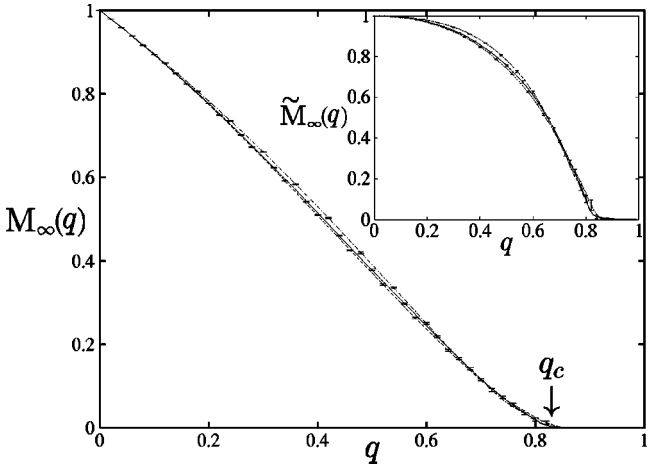

FIG. 3. Fraction of nodes M_∞ in the giant component depending on the fraction q of nodes removed from the network. The graph compares three models at $N=100\,000$ (from top to bottom: Barabási-Albert with initial attractiveness, LR model, and NR model). The theoretical value $q_c=5/6$ is marked by an arrow. The inset shows the fraction $\tilde{M}_\infty$ of the number of nodes in the giant component relative to the remaining number of nodes in the network $(1-q)N$.

the curves. Measuring a second moment of $\langle k^2\rangle \simeq 28$, we find that the simulated networks already correspond to the thermodynamic limit. The assortative NR model clearly shows a percolation transition. Of course, this does not allow any conclusions about networks with assortative correlations other than ours. For the LR model the theoretical value of the percolation threshold calculated according to Eq. (5) is $q_c = 5/6$. Note that the critical exponent for the fraction of nodes in the giant component is $\beta=1$, which is the regular result for infinite-dimensional systems [37], but which is different from the diverging β for the link-randomized BA model (without initial attractiveness). This exponent can clearly be seen in the simulations.

CONCLUSION

Networks grown based on preferential attachment display correlations. These correlations can be changed by applying randomization algorithms. We consider three different models of scale-free networks: the genuine Barabási-Albert construction based on growth and preferential attachment and two networks emerging when randomizing it with respect to links or nodes. We point out that the BA model shows dissortative behavior with respect to the nodes' degrees, while the node-randomized network shows assortative mixing. However, these strong differences in the shell structure lead only to moderate quantitative differences in the percolation behavior of the networks. This same result is found for a network with finite second moment of the degree distribution and its corresponding randomized models.

ACKNOWLEDGMENT

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[3] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comput. Commun. Rev. **29**, 251 (1999).
[4] R. Pastor-Satorras, A. Vazquez, and A. Vespignani, Phys. Rev. Lett. **87**, 258701 (2001).
[5] R. Albert, H. Jeong, and A.-L. Barabasi, Nature (London) **401**, 130 (1999).
[6] A. Broder, R. Kumar, F. Maghoul, P. Raphavan, S. Rajagopalan, R. Stata, A. Tomkins, and J. Wiener, Comput. Netw. **33**, 309 (2000).
[7] D.J. Watts and S.H. Strogatz, Nature (London) **393**, 440 (1998).
[8] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.-L. Barabási, Nature (London) **407**, 651 (2000).
[9] D.A. Fell and A. Wagner, Nat. Biotechnol. **18**, 1121 (2000).
[10] H. Jeong, S.P. Mason, A.-L. Barabási, and Z.N. Oltvai, Nature (London) **411**, 41 (2001).
[11] F. Liljeros, C. Edling, L.A.N. Amaral, H.E. Stanley, and Y. Åberg, Nature (London) **411**, 907 (2001).
[12] M.E.J. Newman, Proc. Natl. Acad. Sci. U.S.A. **98**, 404 (2001).
[13] M.E.J. Newman, Phys. Rev. E **64**, 016131 (2001).
[14] A.L.N. Amaral, M. Barthélémy, and H.E. Stanley, Proc. Natl. Acad. Sci. U.S.A. **97**, 11149 (2000).
[15] R. Albert and A.-L. Barabási, Phys. Rev. Lett. **85**, 5234 (2000).
[16] J. Abello, A. Buchsbaum, and J. Westbrook, Lect. Notes Comput. Sci. **1461**, 332 (1998).
[17] M.E.J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[18] J. Berg and M. Lässig, Phys. Rev. Lett. **89**, 228701 (2002).
[19] V.M. Eguíluz and K. Klemm, Phys. Rev. Lett. **89**, 108701 (2002).
[20] M. Boguñá and R. Pastor-Satorras, Phys. Rev. E **66**, 047104 (2002).
[21] S. Maslov and K. Sneppen, Science **296**, 910 (2002).
[22] A. Vázquez and Y. Moreno, Phys. Rev. E **67**, 015101 (2003).
[23] K.-I. Goh, E. Oh, B. Kahng, and D. Kim, Phys. Rev. E **67**, 017101 (2003).
[24] M. Boguñá, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **90**, 028701 (2003).
[25] M.E.J. Newman, Phys. Rev. E **67**, 026126 (2003).
[26] M.A. Serrano and M. Boguñá, e-print cond-mat/0301015.
[27] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[28] P.L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[29] S.N. Dorogovtsev, J.F.F. Mendes, and A.N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[30] P.L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).
[31] E.A. Bender and E.R. Canfield, J. Comb. Theory, Ser. A **24**, 296 (1978).

[32] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).
[33] C.P. Warren, L.M. Sander, and I.M. Sokolov, Phys. Rev. E **66**, 056105 (2002).
[34] D. ben-Avraham, A.F. Rozenfeld, R. Cohen, and S. Havlin, e-print cond-mat/0301504.
[35] R. Cohen, D. Dolev, S. Havlin, T. Kalisky, O. Mokryn, and Y. Shavitt, Leibniz Center Technical Report No. 2002-49 (unpublished).
[36] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[37] R. Cohen, D. ben-Avraham, and S. Havlin, Phys. Rev. E **66**, 036113 (2002).
[38] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000).
[39] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **86**, 3682 (2001).
[40] D.S. Callaway, M.E.J. Newman, S.H. Strogatz, and D.J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[41] S.N. Dorogovtsev and J.F.F. Mendes, Phys. Rev. E **63**, 056125 (2001).